4/8/26

We ended last time with

<u>Def</u> If V is a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$ then

$$\operatorname{proj}_V(\vec{x}) = \vec{x}^{\parallel} \qquad \text{(orthogonal projection)}$$

$$= (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

This will be part of Gram-Schmidt

for all $\vec{x} \in \mathbb{R}^n$. (This is the formula in terms of $\vec{u}_1, \dots, \vec{u}_m$.)

We still have the issue of finding orthonormal bases.

<u>Ex</u> Let $V \subset \mathbb{R}^4$ be defined as the col. space of $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$

and $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}$.

Find $\operatorname{proj}_V(\vec{x})$.

Turns out an orthonormal basis for V is $\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

$$\operatorname{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2$$

$$= 6\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 2\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

Here's a theorem with a similar flavor.

5.1.6

Thm If $\vec{u}_1, \dots, \vec{u}_n$ is an orthonormal basis of $\mathbb{R}^n$ then

$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$$

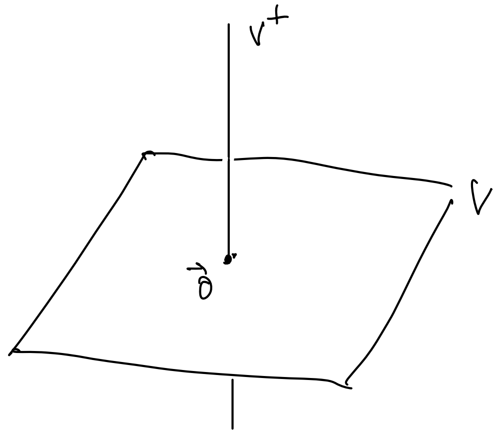
$\forall\, x \in \mathbb{R}^n$.

(skip proof).

So if you have an orthonormal basis, it's easy to write any vector in $\mathbb{R}^n$ as a lin comb of the basis elements. (We know this can always be done with any basis, but "orthonormal" makes it a much simpler computation.)

Def Let V be a subspace of $\mathbb{R}^n$. The orthogonal complement of V is the set

$$V^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{x} = 0 \ \forall\, \vec{v} \in V\}.$$



Important properties (Th. 5.1.8)

1. $V^\perp$ is a subspace of $\mathbb{R}^n$
2. $V \cap V^\perp = \{\vec{0}\}$
3. $\dim V + \dim V^\perp = n$
4. $(V^\perp)^\perp = V$

Note how the theory we've built up helps prove these.

proof

1. Look at the lin transf $proj_V : \mathbb{R}^n \to V$.

   Then $\ker(proj_V) = V^\perp$ so it's a subspace (ker T is always a subspace)

2. If $\vec{x} \in V$ and $\vec{x} \in V^\perp$ then $\vec{x} \cdot \vec{x} = 0$ so $\vec{x} = \vec{0}$

3. Again look at $proj_V : \mathbb{R}^n \to V$.

   $\ker(proj_V) = V^\perp$

   $\mathrm{im}(proj_V) = V$

$\Rightarrow$ By rank-nullity, we're done

4. $(V^\perp)^\perp = \{x \in \mathbb{R}^n \mid \vec{x} \cdot \vec{y} = 0 \ \forall y \in V^\perp\}$

   Step 1 $V \subset (V^\perp)^\perp$

Recall $V^\perp = \{\vec{y} \in \mathbb{R}^n \mid \vec{y} \cdot \vec{x} = 0 \ \forall \vec{x} \in V\}$

   So if $\vec{x} \in V$, $\vec{x} \cdot \vec{y} = 0 \ \forall \vec{y} \in V^\perp$, so done step 1

   Step 2 Show $\dim V = \dim (V^\perp)^\perp$

   If W is a subspace of $\mathbb{R}^n$, we just saw $\dim W + \dim W^\perp = n$ We'll use this twice

① So $\dim V + \dim V^\perp = n \Rightarrow \dim V = n - \dim V^\perp$

OTOH set $W = V^\perp$. ② $\dim V^\perp + \dim (V^\perp)^\perp = n \Rightarrow \dim (V^\perp)^\perp = n - \dim V^\perp$

   Done step 2

But we know if $A \subseteq B$ are subspaces of $\mathbb{R}^n$ and $\dim A = \dim B$ then $A = B$.

Skip the rest of §5.1

§5.2 Gram-Schmidt Process

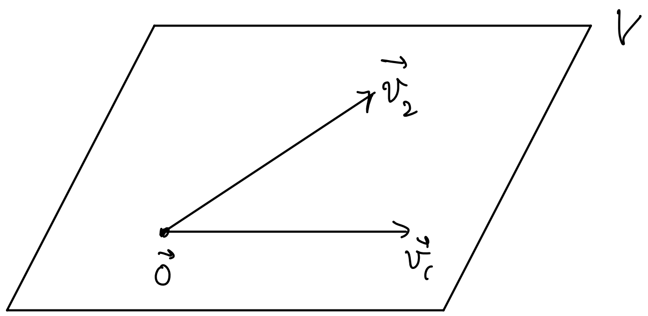
(This is how we get orthonormal bases.)

Easiest case

if $\vec{v} \in \mathbb{R}^1$ then $\left\{ \frac{1}{\|\vec{v}\|} \vec{v} \right\}$ is an orthonormal basis for $\mathbb{R}^1$

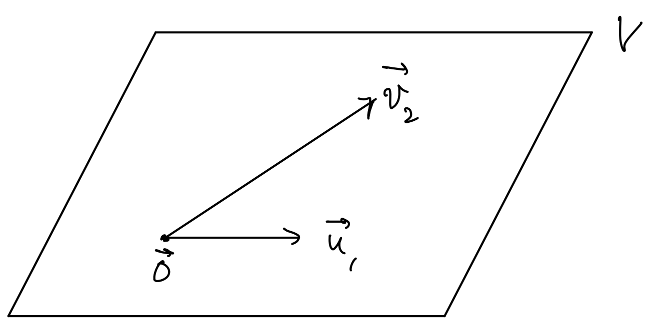
Now look at the case where V is a plane (in $\mathbb{R}^n$). This will show the way in general

We can always start with an arbitrary basis for V, say $\{\vec{v}_1, \vec{v}_2\}$.



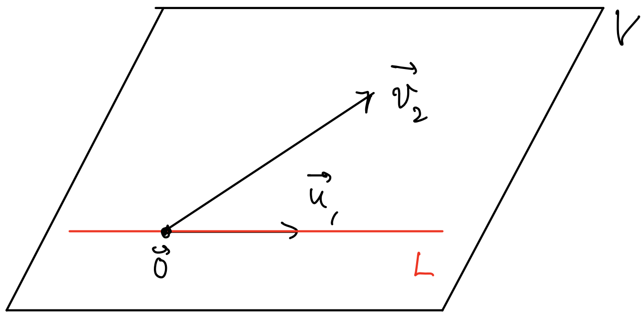
We want an orthonormal basis for V.

Step 1 $\quad \vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1$



Step 2 Find a vector orthogonal to $\vec{u}_1$ w/o worrying for now if it's of unit length. We'll replace $\vec{v}_2$ by this vector, temporarily.

Let L be the line spanned by $\vec{u}_1$



Recall we can decompose $\vec{v}_2$ as

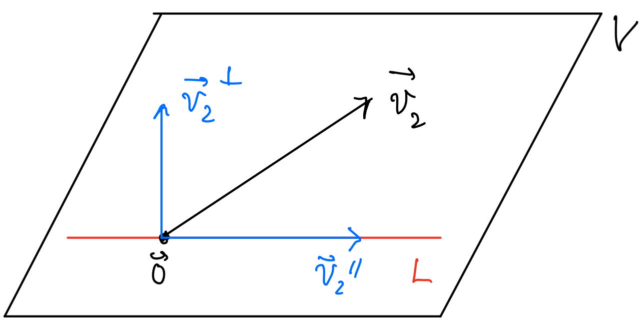
$$\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}$$

where $\underbrace{\vec{v}_2^{\parallel} \text{ is parallel to } L}$ and $\underbrace{\vec{v}_2^{\perp} \text{ is perpendicular to } L}$.

namely $\vec{v}_2^{\parallel} = \text{proj}_L(\vec{v}_2)$

$= (\vec{u}_1 \cdot \vec{v}_2)\,\vec{u}_1$

this is the vector we're looking for in step 2



$\vec{v}_1, \vec{v}_2$ are orthogonal

$\vec{v}_2^{\parallel}$ is not necessarily equal to $\vec{u}_1$.
we don't need that.